

AP Calculus AB/BC Review

Series and Sequences

SOLUTIONS AND SCORING

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2015 SCORING GUIDELINES

Question 6

The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \dots + \frac{(-3)^{n-1}}{n} x^n + \dots$ and converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series.

- (a) Use the ratio test to find R .
- (b) Write the first four nonzero terms of the Maclaurin series for f' , the derivative of f . Express f' as a rational function for $|x| < R$.
- (c) Write the first four nonzero terms of the Maclaurin series for e^x . Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

- (a) Let a_n be the n th term of the Maclaurin series.

$$\frac{a_{n+1}}{a_n} = \frac{(-3)^n x^{n+1}}{n+1} \cdot \frac{n}{(-3)^{n-1} x^n} = \frac{-3n}{n+1} \cdot x$$

$$\lim_{n \rightarrow \infty} \left| \frac{-3n}{n+1} \cdot x \right| = 3|x|$$

$$3|x| < 1 \Rightarrow |x| < \frac{1}{3}$$

The radius of convergence is $R = \frac{1}{3}$.

- (b) The first four nonzero terms of the Maclaurin series for f' are $1 - 3x + 9x^2 - 27x^3$.

$$f'(x) = \frac{1}{1 - (-3x)} = \frac{1}{1 + 3x}$$

- (c) The first four nonzero terms of the Maclaurin series for e^x are $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$.

The product of the Maclaurin series for e^x and the Maclaurin series for f is

$$\begin{aligned} & \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right) \left(x - \frac{3}{2}x^2 + 3x^3 - \dots \right) \\ &= x - \frac{1}{2}x^2 + 2x^3 + \dots \end{aligned}$$

The third-degree Taylor polynomial for $g(x) = e^x f(x)$

about $x = 0$ is $T_3(x) = x - \frac{1}{2}x^2 + 2x^3$.

3 : $\begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{determines radius of convergence} \end{cases}$

3 : $\begin{cases} 2 : \text{first four nonzero terms} \\ 1 : \text{rational function} \end{cases}$

3 : $\begin{cases} 1 : \text{first four nonzero terms} \\ \text{of the Maclaurin series for } e^x \\ 2 : \text{Taylor polynomial} \end{cases}$

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2014 SCORING GUIDELINES

Question 6

The Taylor series for a function f about $x = 1$ is given by $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n} (x-1)^n$ and converges to $f(x)$ for $|x-1| < R$, where R is the radius of convergence of the Taylor series.

- (a) Find the value of R .
- (b) Find the first three nonzero terms and the general term of the Taylor series for f' , the derivative of f , about $x = 1$.
- (c) The Taylor series for f' about $x = 1$, found in part (b), is a geometric series. Find the function f' to which the series converges for $|x-1| < R$. Use this function to determine f for $|x-1| < R$.

- (a) Let a_n be the n th term of the Taylor series.

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{(-1)^{n+2} 2^{n+1} (x-1)^{n+1}}{n+1} \cdot \frac{n}{(-1)^{n+1} 2^n (x-1)^n} \\ &= \frac{-2n(x-1)}{n+1} \end{aligned}$$

$$\lim_{n \rightarrow \infty} \left| \frac{-2n(x-1)}{n+1} \right| = 2|x-1|$$

$$2|x-1| < 1 \Rightarrow |x-1| < \frac{1}{2}$$

The radius of convergence is $R = \frac{1}{2}$.

- (b) The first three nonzero terms are
 $2 - 4(x-1) + 8(x-1)^2$.

The general term is $(-1)^{n+1} 2^n (x-1)^{n-1}$ for $n \geq 1$.

- (c) The common ratio is $-2(x-1)$.

$$f'(x) = \frac{2}{1 - (-2(x-1))} = \frac{2}{2x-1} \text{ for } |x-1| < \frac{1}{2}$$

$$f(x) = \int \frac{2}{2x-1} dx = \ln|2x-1| + C$$

$$f(1) = 0$$

$$\ln|1| + C = 0 \Rightarrow C = 0$$

$$f(x) = \ln|2x-1| \text{ for } |x-1| < \frac{1}{2}$$

3 : $\begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{determines radius of convergence} \end{cases}$

3 : $\begin{cases} 2 : \text{first three nonzero terms} \\ 1 : \text{general term} \end{cases}$

3 : $\begin{cases} 1 : f'(x) \\ 1 : \text{antiderivative} \\ 1 : f(x) \end{cases}$

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Question 6

A function f has derivatives of all orders at $x = 0$. Let $P_n(x)$ denote the n th-degree Taylor polynomial for f about $x = 0$.

- (a) It is known that $f(0) = -4$ and that $P_1\left(\frac{1}{2}\right) = -3$. Show that $f'(0) = 2$.
- (b) It is known that $f''(0) = -\frac{2}{3}$ and $f'''(0) = \frac{1}{3}$. Find $P_3(x)$.
- (c) The function h has first derivative given by $h'(x) = f(2x)$. It is known that $h(0) = 7$. Find the third-degree Taylor polynomial for h about $x = 0$.

(a) $P_1(x) = f(0) + f'(0)x = -4 + f'(0)x$

$$P_1\left(\frac{1}{2}\right) = -4 + f'(0) \cdot \frac{1}{2} = -3$$

$$f'(0) \cdot \frac{1}{2} = 1$$

$$f'(0) = 2$$

$$2 : \begin{cases} 1 : \text{uses } P_1(x) \\ 1 : \text{verifies } f'(0) = 2 \end{cases}$$

(b) $P_3(x) = -4 + 2x + \left(-\frac{2}{3}\right) \cdot \frac{x^2}{2!} + \frac{1}{3} \cdot \frac{x^3}{3!}$

$$= -4 + 2x - \frac{1}{3}x^2 + \frac{1}{18}x^3$$

$$3 : \begin{cases} 1 : \text{first two terms} \\ 1 : \text{third term} \\ 1 : \text{fourth term} \end{cases}$$

- (c) Let $Q_n(x)$ denote the Taylor polynomial of degree n for h about $x = 0$.

$$h'(x) = f(2x) \Rightarrow Q_3'(x) = -4 + 2(2x) - \frac{1}{3}(2x)^2$$

$$Q_3(x) = -4x + 4 \cdot \frac{x^2}{2} - \frac{4}{3} \cdot \frac{x^3}{3} + C; \quad C = Q_3(0) = h(0) = 7$$

$$Q_3(x) = 7 - 4x + 2x^2 - \frac{4}{9}x^3$$

OR

$$h'(x) = f(2x), \quad h''(x) = 2f'(2x), \quad h'''(x) = 4f''(2x)$$

$$h(0) = f(0) = -4, \quad h''(0) = 2f'(0) = 4, \quad h'''(0) = 4f''(0) = -\frac{8}{3}$$

$$Q_3(x) = 7 - 4x + 4 \cdot \frac{x^2}{2!} - \frac{8}{3} \cdot \frac{x^3}{3!} = 7 - 4x + 2x^2 - \frac{4}{9}x^3$$

$$4 : \begin{cases} 2 : \text{applies } h'(x) = f(2x) \\ 1 : \text{constant term} \\ 1 : \text{remaining terms} \end{cases}$$

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Question 6

The function g has derivatives of all orders, and the Maclaurin series for g is

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+3} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \dots$$

- (a) Using the ratio test, determine the interval of convergence of the Maclaurin series for g .
- (b) The Maclaurin series for g evaluated at $x = \frac{1}{2}$ is an alternating series whose terms decrease in absolute value to 0. The approximation for $g\left(\frac{1}{2}\right)$ using the first two nonzero terms of this series is $\frac{17}{120}$. Show that this approximation differs from $g\left(\frac{1}{2}\right)$ by less than $\frac{1}{200}$.
- (c) Write the first three nonzero terms and the general term of the Maclaurin series for $g'(x)$.

(a) $\left| \frac{x^{2n+3}}{2n+5} \cdot \frac{2n+3}{x^{2n+1}} \right| = \left(\frac{2n+3}{2n+5} \right) \cdot x^2$

$$\lim_{n \rightarrow \infty} \left(\frac{2n+3}{2n+5} \right) \cdot x^2 = x^2$$

$$x^2 < 1 \Rightarrow -1 < x < 1$$

The series converges when $-1 < x < 1$.

When $x = -1$, the series is $-\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$

This series converges by the Alternating Series Test.

When $x = 1$, the series is $\frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \dots$

This series converges by the Alternating Series Test.

Therefore, the interval of convergence is $-1 \leq x \leq 1$.

- 5 : $\left\{ \begin{array}{l} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{identifies interior of} \\ \quad \text{interval of convergence} \\ 1 : \text{considers both endpoints} \\ 1 : \text{analysis and interval of convergence} \end{array} \right.$

(b) $\left| g\left(\frac{1}{2}\right) - \frac{17}{120} \right| < \frac{\left(\frac{1}{2}\right)^5}{7} = \frac{1}{224} < \frac{1}{200}$

- 2 : $\left\{ \begin{array}{l} 1 : \text{uses the third term as an error bound} \\ 1 : \text{error bound} \end{array} \right.$

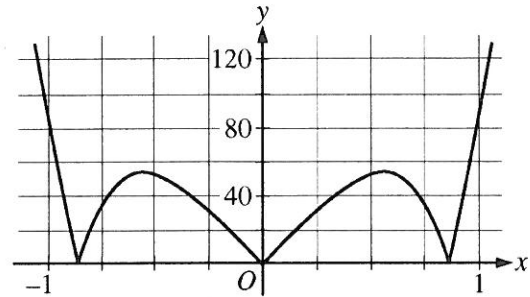
(c) $g'(x) = \frac{1}{3} - \frac{3}{5}x^2 + \frac{5}{7}x^4 + \dots + (-1)^n \left(\frac{2n+1}{2n+3} \right) x^{2n} + \dots$

- 2 : $\left\{ \begin{array}{l} 1 : \text{first three terms} \\ 1 : \text{general term} \end{array} \right.$

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2011 SCORING GUIDELINES

Question 6

Let $f(x) = \sin(x^2) + \cos x$. The graph of $y = |f^{(5)}(x)|$ is shown above.



Graph of $y = |f^{(5)}(x)|$

- (a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin(x^2)$ about $x = 0$.
- (b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin(x^2)$, found in part (a), to write the first four nonzero terms of the Taylor series for f about $x = 0$.
- (c) Find the value of $f^{(6)}(0)$.
- (d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Using information from the graph of $y = |f^{(5)}(x)|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.

(a) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots$

3 : $\begin{cases} 1 : \text{series for } \sin x \\ 2 : \text{series for } \sin(x^2) \end{cases}$

(b) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

$f(x) = 1 + \frac{x^2}{2} + \frac{x^4}{4!} - \frac{121x^6}{6!} + \dots$

3 : $\begin{cases} 1 : \text{series for } \cos x \\ 2 : \text{series for } f(x) \end{cases}$

- (c) $\frac{f^{(6)}(0)}{6!}$ is the coefficient of x^6 in the Taylor series for f about $x = 0$. Therefore $f^{(6)}(0) = -121$.

1 : answer

- (d) The graph of $y = |f^{(5)}(x)|$ indicates that $\max_{0 \leq x \leq \frac{1}{4}} |f^{(5)}(x)| < 40$.

Therefore

$$\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| \leq \frac{\max_{0 \leq x \leq \frac{1}{4}} |f^{(5)}(x)|}{5!} \cdot \left(\frac{1}{4}\right)^5 < \frac{40}{120 \cdot 4^5} = \frac{1}{3072} < \frac{1}{3000}.$$

2 : $\begin{cases} 1 : \text{form of the error bound} \\ 1 : \text{analysis} \end{cases}$

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2011 SCORING GUIDELINES (Form B)

Question 6

Let $f(x) = \ln(1 + x^3)$.

- (a) The Maclaurin series for $\ln(1 + x)$ is $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1} \cdot \frac{x^n}{n} + \cdots$. Use the series to write the first four nonzero terms and the general term of the Maclaurin series for f .
- (b) The radius of convergence of the Maclaurin series for f is 1. Determine the interval of convergence. Show the work that leads to your answer.
- (c) Write the first four nonzero terms of the Maclaurin series for $f'(t^2)$. If $g(x) = \int_0^x f'(t^2) dt$, use the first two nonzero terms of the Maclaurin series for g to approximate $g(1)$.
- (d) The Maclaurin series for g , evaluated at $x = 1$, is a convergent alternating series with individual terms that decrease in absolute value to 0. Show that your approximation in part (c) must differ from $g(1)$ by less than $\frac{1}{5}$.

(a) $x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \cdots + (-1)^{n+1} \cdot \frac{x^{3n}}{n} + \cdots$

- (b) The interval of convergence is centered at $x = 0$.
 At $x = -1$, the series is $-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots - \frac{1}{n} - \cdots$, which diverges because the harmonic series diverges.

At $x = 1$, the series is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n+1} \cdot \frac{1}{n} + \cdots$, the alternating harmonic series, which converges.

Therefore the interval of convergence is $-1 < x \leq 1$.

- (c) The Maclaurin series for $f'(x)$, $f'(t^2)$, and $g(x)$ are

$$f'(x) : \sum_{n=1}^{\infty} (-1)^{n+1} \cdot 3x^{3n-1} = 3x^2 - 3x^5 + 3x^8 - 3x^{11} + \cdots$$

$$f'(t^2) : \sum_{n=1}^{\infty} (-1)^{n+1} \cdot 3t^{6n-2} = 3t^4 - 3t^{10} + 3t^{16} - 3t^{22} + \cdots$$

$$g(x) : \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{3x^{6n-1}}{6n-1} = \frac{3x^5}{5} - \frac{3x^{11}}{11} + \frac{3x^{17}}{17} - \frac{3x^{23}}{23} + \cdots$$

$$\text{Thus } g(1) \approx \frac{3}{5} - \frac{3}{11} = \frac{18}{55}.$$

- (d) The Maclaurin series for g evaluated at $x = 1$ is alternating, and the terms decrease in absolute value to 0.

$$\text{Thus } \left| g(1) - \frac{18}{55} \right| < \frac{3 \cdot 1^{17}}{17} = \frac{3}{17} < \frac{1}{5}.$$

2 : $\begin{cases} 1 : \text{first four terms} \\ 1 : \text{general term} \end{cases}$

2 : answer with analysis

4 : $\begin{cases} 1 : \text{two terms for } f'(t^2) \\ 1 : \text{other terms for } f'(t^2) \\ 1 : \text{first two terms for } g(x) \\ 1 : \text{approximation} \end{cases}$

1 : analysis

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Question 6

$$f(x) = \begin{cases} \frac{\cos x - 1}{x^2} & \text{for } x \neq 0 \\ -\frac{1}{2} & \text{for } x = 0 \end{cases}$$

The function f , defined above, has derivatives of all orders. Let g be the function defined by

$$g(x) = 1 + \int_0^x f(t) \, dt.$$

- (a) Write the first three nonzero terms and the general term of the Taylor series for $\cos x$ about $x = 0$. Use this series to write the first three nonzero terms and the general term of the Taylor series for f about $x = 0$.
- (b) Use the Taylor series for f about $x = 0$ found in part (a) to determine whether f has a relative maximum, relative minimum, or neither at $x = 0$. Give a reason for your answer.
- (c) Write the fifth-degree Taylor polynomial for g about $x = 0$.
- (d) The Taylor series for g about $x = 0$, evaluated at $x = 1$, is an alternating series with individual terms that decrease in absolute value to 0. Use the third-degree Taylor polynomial for g about $x = 0$ to estimate the value of $g(1)$. Explain why this estimate differs from the actual value of $g(1)$ by less than $\frac{1}{6!}$.

(a) $\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$

$$f(x) = -\frac{1}{2} + \frac{x^2}{4!} - \frac{x^4}{6!} + \cdots + (-1)^{n+1} \frac{x^{2n}}{(2n+2)!} + \cdots$$

$$3 : \begin{cases} 1 : \text{terms for } \cos x \\ 2 : \text{terms for } f \\ 1 : \text{first three terms} \\ 1 : \text{general term} \end{cases}$$

- (b) $f'(0)$ is the coefficient of x in the Taylor series for f about $x = 0$, so $f'(0) = 0$.

$$\frac{f''(0)}{2!} = \frac{1}{4!} \text{ is the coefficient of } x^2 \text{ in the Taylor series for } f \text{ about}$$

$$x = 0, \text{ so } f''(0) = \frac{1}{12}.$$

Therefore, by the Second Derivative Test, f has a relative minimum at $x = 0$.

$$2 : \begin{cases} 1 : \text{determines } f'(0) \\ 1 : \text{answer with reason} \end{cases}$$

(c) $P_5(x) = 1 - \frac{x}{2} + \frac{x^3}{3 \cdot 4!} - \frac{x^5}{5 \cdot 6!}$

$$2 : \begin{cases} 1 : \text{two correct terms} \\ 1 : \text{remaining terms} \end{cases}$$

(d) $g(1) \approx 1 - \frac{1}{2} + \frac{1}{3 \cdot 4!} = \frac{37}{72}$

Since the Taylor series for g about $x = 0$ evaluated at $x = 1$ is alternating and the terms decrease in absolute value to 0, we know

$$\left| g(1) - \frac{37}{72} \right| < \frac{1}{5 \cdot 6!} < \frac{1}{6!}.$$

$$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{explanation} \end{cases}$$

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2010 SCORING GUIDELINES (Form B)

Question 6

The Maclaurin series for the function f is given by $f(x) = \sum_{n=2}^{\infty} \frac{(-1)^n (2x)^n}{n-1}$ on its interval of convergence.

(a) Find the interval of convergence for the Maclaurin series of f . Justify your answer.

(b) Show that $y = f(x)$ is a solution to the differential equation $xy' - y = \frac{4x^2}{1+2x}$ for $|x| < R$, where R is the radius of convergence from part (a).

$$(a) \lim_{n \rightarrow \infty} \left| \frac{\frac{(2x)^{n+1}}{(n+1)-1}}{\frac{(2x)^n}{n-1}} \right| = \lim_{n \rightarrow \infty} \left| 2x \cdot \frac{n-1}{n} \right| = \lim_{n \rightarrow \infty} \left| 2x \cdot \frac{n-1}{n} \right| = |2x|$$

$$|2x| < 1 \text{ for } |x| < \frac{1}{2}$$

Therefore the radius of convergence is $\frac{1}{2}$.

$$\text{When } x = -\frac{1}{2}, \text{ the series is } \sum_{n=2}^{\infty} \frac{(-1)^n (-1)^n}{n-1} = \sum_{n=2}^{\infty} \frac{1}{n-1}.$$

This is the harmonic series, which diverges.

$$\text{When } x = \frac{1}{2}, \text{ the series is } \sum_{n=2}^{\infty} \frac{(-1)^n 1^n}{n-1} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n-1}.$$

This is the alternating harmonic series, which converges.

The interval of convergence for the Maclaurin series of f is $\left(-\frac{1}{2}, \frac{1}{2}\right]$.

$$(b) y = \frac{(2x)^2}{1} - \frac{(2x)^3}{2} + \frac{(2x)^4}{3} - \dots + \frac{(-1)^n (2x)^n}{n-1} + \dots$$

$$= 4x^2 - 4x^3 + \frac{16}{3}x^4 - \dots + \frac{(-1)^n (2x)^n}{n-1} + \dots$$

$$y' = 8x - 12x^2 + \frac{64}{3}x^3 - \dots + \frac{(-1)^n n(2x)^{n-1} \cdot 2}{n-1} + \dots$$

$$xy' = 8x^2 - 12x^3 + \frac{64}{3}x^4 - \dots + \frac{(-1)^n n(2x)^n}{n-1} + \dots$$

$$xy' - y = 4x^2 - 8x^3 + 16x^4 - \dots + (-1)^n (2x)^n + \dots$$

$$= 4x^2(1 - 2x + 4x^2 - \dots + (-1)^n (2x)^{n-2} + \dots)$$

The series $1 - 2x + 4x^2 - \dots + (-1)^n (2x)^{n-2} + \dots = \sum_{n=0}^{\infty} (-2x)^n$ is a

geometric series that converges to $\frac{1}{1+2x}$ for $|x| < \frac{1}{2}$. Therefore

$$xy' - y = 4x^2 \cdot \frac{1}{1+2x} \text{ for } |x| < \frac{1}{2}.$$

5 : { 1 : sets up ratio
1 : limit evaluation
1 : radius of convergence
1 : considers both endpoints
1 : analysis and interval of convergence

4 : { 1 : series for y'
1 : series for xy'
1 : series for $xy' - y$
1 : analysis with geometric series

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2009 SCORING GUIDELINES

Question 6

The Maclaurin series for e^x is $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots$. The continuous function f is defined by $f(x) = \frac{e^{(x-1)^2} - 1}{(x-1)^2}$ for $x \neq 1$ and $f(1) = 1$. The function f has derivatives of all orders at $x = 1$.

- Write the first four nonzero terms and the general term of the Taylor series for $e^{(x-1)^2}$ about $x = 1$.
- Use the Taylor series found in part (a) to write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.
- Use the ratio test to find the interval of convergence for the Taylor series found in part (b).
- Use the Taylor series for f about $x = 1$ to determine whether the graph of f has any points of inflection.

(a) $1 + (x-1)^2 + \frac{(x-1)^4}{2} + \frac{(x-1)^6}{6} + \cdots + \frac{(x-1)^{2n}}{n!} + \cdots$

2 : $\begin{cases} 1 : \text{first four terms} \\ 1 : \text{general term} \end{cases}$

(b) $1 + \frac{(x-1)^2}{2} + \frac{(x-1)^4}{6} + \frac{(x-1)^6}{24} + \cdots + \frac{(x-1)^{2n}}{(n+1)!} + \cdots$

2 : $\begin{cases} 1 : \text{first four terms} \\ 1 : \text{general term} \end{cases}$

(c) $\lim_{n \rightarrow \infty} \left| \frac{\frac{(x-1)^{2n+2}}{(n+2)!}}{\frac{(x-1)^{2n}}{(n+1)!}} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)!}{(n+2)!} (x-1)^2 = \lim_{n \rightarrow \infty} \frac{(x-1)^2}{n+2} = 0$

Therefore, the interval of convergence is $(-\infty, \infty)$.

3 : $\begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{answer} \end{cases}$

(d) $f'''(x) = 1 + \frac{4 \cdot 3}{6}(x-1)^2 + \frac{6 \cdot 5}{24}(x-1)^4 + \cdots$
 $+ \frac{2n(2n-1)}{(n+1)!}(x-1)^{2n-2} + \cdots$

2 : $\begin{cases} 1 : f'''(x) \\ 1 : \text{answer} \end{cases}$

Since every term of this series is nonnegative, $f'''(x) \geq 0$ for all x .
 Therefore, the graph of f has no points of inflection.

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2009 SCORING GUIDELINES (Form B)

Question 6

The function f is defined by the power series

$$f(x) = 1 + (x+1) + (x+1)^2 + \cdots + (x+1)^n + \cdots = \sum_{n=0}^{\infty} (x+1)^n$$

for all real numbers x for which the series converges.

- (a) Find the interval of convergence of the power series for f . Justify your answer.
- (b) The power series above is the Taylor series for f about $x = -1$. Find the sum of the series for f .
- (c) Let g be the function defined by $g(x) = \int_{-1}^x f(t) dt$. Find the value of $g\left(-\frac{1}{2}\right)$, if it exists, or explain why $g\left(-\frac{1}{2}\right)$ cannot be determined.
- (d) Let h be the function defined by $h(x) = f(x^2 - 1)$. Find the first three nonzero terms and the general term of the Taylor series for h about $x = 0$, and find the value of $h\left(\frac{1}{2}\right)$.

- (a) The power series is geometric with ratio $(x+1)$.
 The series converges if and only if $|x+1| < 1$.
 Therefore, the interval of convergence is $-2 < x < 0$.

OR

$$\lim_{n \rightarrow \infty} \left| \frac{(x+1)^{n+1}}{(x+1)^n} \right| = |x+1| < 1 \text{ when } -2 < x < 0$$

At $x = -2$, the series is $\sum_{n=0}^{\infty} (-1)^n$, which diverges since the

terms do not converge to 0. At $x = 0$, the series is $\sum_{n=0}^{\infty} 1$,

which similarly diverges. Therefore, the interval of convergence is $-2 < x < 0$.

- (b) Since the series is geometric,

$$f(x) = \sum_{n=0}^{\infty} (x+1)^n = \frac{1}{1-(x+1)} = -\frac{1}{x} \text{ for } -2 < x < 0.$$

(c) $g\left(-\frac{1}{2}\right) = \int_{-1}^{-\frac{1}{2}} -\frac{1}{x} dx = -\ln|x| \Big|_{x=-1}^{x=-\frac{1}{2}} = \ln 2$

(d) $h(x) = f(x^2 - 1) = 1 + x^2 + x^4 + \cdots + x^{2n} + \cdots$

$$h\left(\frac{1}{2}\right) = f\left(-\frac{3}{4}\right) = \frac{4}{3}$$

$$3 : \begin{cases} 1 : \text{identifies as geometric} \\ 1 : |x+1| < 1 \\ 1 : \text{interval of convergence} \end{cases}$$

OR

$$3 : \begin{cases} 1 : \text{sets up limit of ratio} \\ 1 : \text{radius of convergence} \\ 1 : \text{interval of convergence} \end{cases}$$

1 : answer

$$2 : \begin{cases} 1 : \text{antiderivative} \\ 1 : \text{value} \end{cases}$$

$$3 : \begin{cases} 1 : \text{first three terms} \\ 1 : \text{general term} \\ 1 : \text{value of } h\left(\frac{1}{2}\right) \end{cases}$$

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Question 3

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
1	11	30	42	99	18
2	80	128	$\frac{488}{3}$	$\frac{448}{3}$	$\frac{584}{9}$
3	317	$\frac{753}{2}$	$\frac{1383}{4}$	$\frac{3483}{16}$	$\frac{1125}{16}$

Let h be a function having derivatives of all orders for $x > 0$. Selected values of h and its first four derivatives are indicated in the table above. The function h and these four derivatives are increasing on the interval $1 \leq x \leq 3$.

- (a) Write the first-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$. Is this approximation greater than or less than $h(1.9)$? Explain your reasoning.
- (b) Write the third-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$.
- (c) Use the Lagrange error bound to show that the third-degree Taylor polynomial for h about $x = 2$ approximates $h(1.9)$ with error less than 3×10^{-4} .

(a) $P_1(x) = 80 + 128(x - 2)$, so $h(1.9) \approx P_1(1.9) = 67.2$

$P_1(1.9) < h(1.9)$ since h' is increasing on the interval $1 \leq x \leq 3$.

4 : $\begin{cases} 2 : P_1(x) \\ 1 : P_1(1.9) \\ 1 : P_1(1.9) < h(1.9) \text{ with reason} \end{cases}$

(b) $P_3(x) = 80 + 128(x - 2) + \frac{488}{6}(x - 2)^2 + \frac{448}{18}(x - 2)^3$

$h(1.9) \approx P_3(1.9) = 67.988$

3 : $\begin{cases} 2 : P_3(x) \\ 1 : P_3(1.9) \end{cases}$

(c) The fourth derivative of h is increasing on the interval

$1 \leq x \leq 3$, so $\max_{1.9 \leq x \leq 2} |h^{(4)}(x)| = \frac{584}{9}$.

Therefore, $|h(1.9) - P_3(1.9)| \leq \frac{584}{9} \frac{|1.9 - 2|^4}{4!}$
 $= 2.7037 \times 10^{-4}$
 $< 3 \times 10^{-4}$

2 : $\begin{cases} 1 : \text{form of Lagrange error estimate} \\ 1 : \text{reasoning} \end{cases}$

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Question 6

Let f be the function given by $f(x) = \frac{2x}{1+x^2}$.

- (a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.
- (b) Does the series found in part (a), when evaluated at $x = 1$, converge to $f(1)$? Explain why or why not.
- (c) The derivative of $\ln(1+x^2)$ is $\frac{2x}{1+x^2}$. Write the first four nonzero terms of the Taylor series for $\ln(1+x^2)$ about $x = 0$.
- (d) Use the series found in part (c) to find a rational number A such that $\left|A - \ln\left(\frac{5}{4}\right)\right| < \frac{1}{100}$. Justify your answer.

(a)
$$\frac{1}{1-u} = 1 + u + u^2 + \cdots + u^n + \cdots$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-x^2)^n + \cdots$$

$$\frac{2x}{1+x^2} = 2x - 2x^3 + 2x^5 - 2x^7 + \cdots + (-1)^n 2x^{2n+1} + \cdots$$

3 : $\begin{cases} 1 : \text{two of the first four terms} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \end{cases}$

- (b) No, the series does not converge when $x = 1$ because when $x = 1$, the terms of the series do not converge to 0.

1 : answer with reason

(c)
$$\ln(1+x^2) = \int_0^x \frac{2t}{1+t^2} dt$$

$$= \int_0^x (2t - 2t^3 + 2t^5 - 2t^7 + \cdots) dt$$

$$= x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \frac{1}{4}x^8 + \cdots$$

2 : $\begin{cases} 1 : \text{two of the first four terms} \\ 1 : \text{remaining terms} \end{cases}$

(d)
$$\ln\left(\frac{5}{4}\right) = \ln\left(1 + \frac{1}{4}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2}\left(\frac{1}{2}\right)^4 + \frac{1}{3}\left(\frac{1}{2}\right)^6 - \frac{1}{4}\left(\frac{1}{2}\right)^8 + \cdots$$

Let $A = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^4 = \frac{7}{32}$.

Since the series is a converging alternating series and the absolute values of the individual terms decrease to 0,

$$\left|A - \ln\left(\frac{5}{4}\right)\right| < \left|\frac{1}{3}\left(\frac{1}{2}\right)^6\right| = \frac{1}{3} \cdot \frac{1}{64} < \frac{1}{100}.$$

3 : $\begin{cases} 1 : \text{uses } x = \frac{1}{2} \\ 1 : \text{value of } A \\ 1 : \text{justification} \end{cases}$